

# Integrable and Superintegrable Klein-Gordon and Schrödinger Type Dimers

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## **Abstract:**

A  $PT$ -symmetric dimer is a two-site nonlinear oscillator or a nonlinear Schrödinger dimer where one site loses and the other site gains energy at the same rate. We present a wide class of integrable oscillator type dimers whose Hamiltonian is of arbitrary even order. Further, we also present a wide class of integrable and superintegrable nonlinear Schrödinger type dimers where again the Hamiltonian is of arbitrary even order.

# 1 Introduction

In recent years, parity-time reversal or  $PT$ -symmetry [1] has received widespread attention from both the physics and mathematics community. While the original application of  $PT$ -symmetry was in quantum mechanics, soon researchers realized [2] that optics can provide a fertile ground to test some of these ideas experimentally [3]. It was realized there that ubiquitous loss can be countered by an overwhelming gain in order to create a  $PT$ -symmetric setup. For example, such a setup in wave guide dimers paved the way for numerous developments especially in the context of nonlinear systems. Not surprisingly, most of the activity is centered around nonlinear Schrödinger type systems. This is understandable since Schrödinger model is on a similar footing as paraxial approximation in the Maxwell equations. However, recently, motivated by experimental activity [4], for example, in the areas of electronic circuits [5], whispering gallery modes, micro-cavities [6], etc., significant theoretical progress has been made. Some recent theoretical activity has also centered on the oscillator type dimers [7].

In a recent publication, Barashenkov et al. [8] have written down integrable Schrödinger type dimer models with a quartic Hamiltonian. The importance of such integrable dimers cannot be overemphasized. They can act as prototype systems (analogous to integrable models in classical mechanics). Their work raises several questions. For example, are there oscillator-type integrable dimer models? And if they exist, then within the rotating wave approximation (RWA), do they lead to integrable Schrödinger type dimer models? Can one generalize and obtain the integrable oscillator as well as Schrödinger type dimers in case the Hamiltonian is of arbitrary even order? And finally, are there superintegrable dimer models? By superintegrable we mean that the system has  $2n - 1$  constants of motion in involution where  $n$  is the degrees of freedom of the classical system (so that the phase space is  $2n$  dimensional) [9]. Note that for an integrable system there are  $n$  constants of motion in involution.

The purpose of this paper is to answer many of the questions raised here. In particular, we present a class of integrable oscillator type dimer models where the Hamiltonian is of arbitrary even order. We further show that within the RWA, these integrable dimers lead to superintegrable Schrödinger type dimers. We generalize and write down the most general superintegrable Schrödinger type dimers where the Hamiltonian is of arbitrary even order. Finally, we also present a class of integrable (but not superintegrable) Schrödinger

type dimers where the Hamiltonian is of arbitrary even order.

## 2 Oscillator Type Integrable Dimers

To motivate the discussion, let us start from the following coupled equations

$$\ddot{u} = -u + kv + \gamma\dot{u} + \epsilon v^3 + \delta v u^2, \quad (1)$$

$$\ddot{v} = -v + ku - \gamma\dot{v} + \epsilon u^3 + \delta u v^2. \quad (2)$$

We now show that in the special case when  $k = \epsilon = 0$ , it is an integrable system. It is easy to see that these coupled equations can be derived from the Hamiltonian

$$H = p_u p_v + \frac{\gamma}{2}(u p_u - v p_v) + (1 - \frac{\gamma^2}{4})uv - \frac{k}{2}(u^2 + v^2) - \frac{\epsilon}{4}(u^4 + v^4) - \frac{\delta}{2}u^2 v^2. \quad (3)$$

In this case the momenta are given by:  $p_u = \dot{v} + \gamma v/2$ ,  $p_v = \dot{u} - \gamma u/2$ . It is remarkable that for arbitrary  $k, \epsilon$  and  $\delta$ , this is a Hamiltonian system.

Remarkably, when  $k = \epsilon = 0$  but arbitrary  $\delta$ , there is another constant of motion

$$C = u\dot{v} - v\dot{u} + \gamma uv. \quad (4)$$

Thus coupled Eqs. (1) and (2) with  $k = \epsilon = 0$  is an integrable PT-invariant oscillator type dimer system with two constants of motion  $H$  and  $C$ . As far as we are aware of, this is the first known PT-invariant, integrable nonlinear oscillator dimer system. This discussion can be immediately generalized and one can obtain a wide class of PT-invariant integrable Klein-Gordon (KG) dimer systems.

In particular, let us consider the following coupled PT-invariant KG dimer system

$$\ddot{u} = -u + kv + \gamma\dot{u} + \epsilon v^{2n+1} + \delta v^n u^{n+1}, \quad n = 1, 2, 3, \dots, \quad (5)$$

$$\ddot{v} = -v + ku - \gamma\dot{v} + \epsilon u^{2n+1} + \delta u^n v^{n+1}. \quad (6)$$

It is easy to see that these coupled equations can be derived from the Hamiltonian

$$H = p_u p_v + \frac{\gamma}{2}(u p_u - v p_v) + (1 - \frac{\gamma^2}{4})uv - \frac{k}{2}(u^2 + v^2) - \frac{\epsilon}{2n+2}(u^{2n+2} + v^{2n+2}) - \frac{\delta}{n+1}u^{n+1}v^{n+1}. \quad (7)$$

In this case,  $p_u = \dot{v} + \gamma v/2$ ,  $p_v = \dot{u} - \gamma u/2$ . It is remarkable that for arbitrary  $k, \epsilon$  and  $\delta$ , this is a Hamiltonian system. It is easy to check that in the case  $\epsilon = k = 0$  but  $\delta$  arbitrary, there is another constant of motion given by Eq. (4).

Thus coupled Eqs. (5) and (6) with  $k = \epsilon = 0$  but arbitrary  $\delta$  and  $n$  is an integrable PT-invariant KG dimer system with two constants of motion  $H$  and  $C$  for arbitrary positive integer values of  $n$ .

One can even further generalize and have an even wider class of PT-invariant integrable oscillator type dimer systems. In particular, consider the following coupled equations

$$\begin{aligned}\ddot{u} = & -u + kv + \gamma\dot{u} + \epsilon_1 v^{2n_1+1} + \epsilon_2 v^{2n_2+1} \\ & + \delta_1 v^{n_1} u^{n_1+1} + \delta_2 v^{n_2} u^{n_2+1},\end{aligned}\tag{8}$$

$$\begin{aligned}\ddot{v} = & -v + ku - \gamma\dot{v} + \epsilon_1 u^{2n_1+1} + \epsilon_2 u^{2n_2+1} \\ & + \delta_1 u^{n_1} v^{n_1+1} + \delta_2 u^{n_2} v^{n_2+1}.\end{aligned}\tag{9}$$

These coupled equations can be derived from the Hamiltonian

$$\begin{aligned}H_2 = & p_u p_v + \frac{\gamma}{2}(u p_u - v p_v) + (1 - \frac{\gamma^2}{4})uv - \frac{k}{2}(u^2 + v^2) - \frac{\epsilon_1}{2n_1+2}(u^{2n_1+2} + v^{2n_1+2}) \\ & - \frac{\epsilon_2}{2n_2+2}(u^{2n_2+2} + v^{2n_2+2}) - \frac{\delta_1}{n_1+1}u^{n_1+1}v^{n_1+1} - \frac{\delta_2}{n_2+1}u^{n_2+1}v^{n_2+1}.\end{aligned}\tag{10}$$

In this case,  $p_u = \dot{v} + \gamma v/2$ ,  $p_v = \dot{u} - \gamma u/2$ . It is easy to check that in case  $\epsilon_{1,2} = k = 0$  but  $\delta_{1,2}$  arbitrary, then  $C$  as defined by Eq. (4) continues to be another constant of motion.

Thus coupled Eqs. (8) and (9) with  $k = \epsilon_{1,2} = 0$  is an integrable PT-invariant KG dimer system with two constants of motion  $H_2$  and  $C$  for arbitrary positive integer values of  $n_1, n_2$  and for arbitrary values of  $\delta_{1,2}$ . Generalization to sum of several terms of the form  $\epsilon_i v^{n_i}$  and  $\delta_i v^{(n_i-1)/2} u^{(n_i+1)/2}$  in  $\ddot{u}$  and appropriate terms in  $\ddot{v}$  is straightforward.

Summarizing, we have obtained a wide class of PT-invariant integrable oscillator type dimer systems. It would thus be worthwhile to study these coupled models in detail. We hope to do that in the near future.

### 3 Superintegrable Schrödinger Type Dimers

We now show that unlike the oscillator type dimers, one can in fact construct a  $n + 1$  parameter family of superintegrable Schrödinger type dimers corresponding to an arbitrary even order Hamiltonian  $H$ .

To motivate the discussion, we first write down the 4th order Hamiltonian and show that it is superintegrable and then generalize the discussion to arbitrary even order.

Consider the following quartic cross-gradient Hamiltonian (which gives rise to cubic dimer equations)

$$H = A(\phi_1^* \phi_2 + \phi_2^* \phi_1) + i\Gamma(\phi_2^* \phi_1 - \phi_1^* \phi_2) + D_1(\phi_1^* \phi_2 + \phi_2^* \phi_1)^2 + D_2|\phi_1|^2|\phi_2|^2. \quad (11)$$

As noted, this gives rise to the cubic dimer equations

$$i\frac{d\phi_1}{dt} = A\phi_1 + i\gamma\phi_1 + 2D_1(\phi_1^* \phi_2 + \phi_2^* \phi_1)\phi_1 + D_2|\phi_1|^2\phi_2, \quad (12)$$

$$i\frac{d\phi_2}{dt} = A\phi_2 - i\gamma\phi_2 + 2D_1(\phi_1^* \phi_2 + \phi_2^* \phi_1)\phi_2 + D_2|\phi_2|^2\phi_1. \quad (13)$$

Notice that in this case the canonical coordinate-momentum pairs are  $\phi_1, \dot{\phi}_2^*$  and  $\phi_2, \dot{\phi}_1^*$  so that in this case the Hamilton's equations are

$$i\frac{d\phi_1}{dt} = \frac{\partial H}{\partial \phi_2^*}, \quad i\frac{d\phi_2}{dt} = \frac{\partial H}{\partial \phi_1^*}. \quad (14)$$

It may be noted here that in order to distinguish the oscillator type and Schrödinger type dimers, we use different symbols  $\phi_1, \phi_2$  instead of  $u, v$  which we used for the oscillator type dimers.

In order to study the question of integrability of the Schrödinger type dimers, we introduce the Stokes variables

$$X = \phi_1^* \phi_2 + \phi_2^* \phi_1, \quad Y = i(\phi_1^* \phi_2 - \phi_2^* \phi_1), \quad Z = |\phi_1|^2 - |\phi_2|^2, \quad R = |\phi_1|^2 + |\phi_2|^2. \quad (15)$$

Using the field equations, it is easy to check that the variables  $X, Y$  as defined by Eq. (15) are time independent, i.e.  $\dot{X} = \dot{Y} = 0$ . Thus, this dimer system is superintegrable with the three constants of motion being  $H, X, Y$  as defined by Eqs. (11) and (15).

Generalization to arbitrary even order is now straightforward. In particular, so long as the cross-gradient Hamiltonian only consists of the arbitrary combinations of the two invariants  $X$  and  $|\phi_1|^2|\phi_2|^2$ ,

then it will always be superintegrable with the three constants of motion being  $H, X, Y$ . For example, the most general  $4n$ 'th order cross-gradient superintegrable Hamiltonian has  $n + 1$  arbitrary parameters  $D_i$  and is given by

$$H = A(\phi_1^* \phi_2 + \phi_2^* \phi_1) + i\gamma(\phi_2^* \phi_1 - \phi_1^* \phi_2) \\ + D_1(\phi_1^* \phi_2 + \phi_2^* \phi_1)^{2n} + D_2|\phi_1|^2|\phi_2|^2(\phi_1^* \phi_2 + \phi_2^* \phi_1)^{2n-2} + \dots + D_{n+1}|\phi_1|^{2n}|\phi_2|^{2n}. \quad (16)$$

Similarly, the most general  $(4n + 2)$ 'th order cross-gradient superintegrable Hamiltonian too has  $n + 1$  arbitrary parameters  $D_i$  and is given by

$$H = A(\phi_1^* \phi_2 + \phi_2^* \phi_1) + i\gamma(\phi_2^* \phi_1 - \phi_1^* \phi_2) \\ + D_1(\phi_1^* \phi_2 + \phi_2^* \phi_1)^{2n+1} + D_2|\phi_1|^2|\phi_2|^2(\phi_1^* \phi_2 + \phi_2^* \phi_1)^{2n-1} + \dots + D_{n+1}|\phi_1|^{2n}|\phi_2|^{2n}(\phi_1^* \phi_2 + \phi_2^* \phi_1). \quad (17)$$

For both these cases using Eqs. (14) it is straightforward to write equations of motion and show that  $\dot{X} = \dot{Y} = 0$ .

It is worth pointing out that if one considers the integrable oscillator type dimer models discussed in the previous section, within the rotating wave approximation, then they in fact lead to superintegrable Schrödinger type dimer models as discussed here with the three constants of motion being cross-gradient  $H, X$  and  $Y$ .

## 4 Wide Class of Schrödinger Type Integrable (but not Superintegrable)

### Dimer Models Based on Cross-Gradient Hamiltonians

We have thus presented a  $n + 1$  parameter family of superintegrable, Schrödinger type dimers with even order cross-gradient Hamiltonians of order  $4n$  or  $4n + 2$ . It is then natural to ask if we can also obtain integrable (but not superintegrable) dimer models with arbitrary even order cross-gradient Hamiltonians. We now show that this is indeed possible and that we get either a  $(n + 1)^2$  or a  $(n + 1)(n + 2)$  parameter family of cross-gradient Hamiltonians of order  $4n$  or  $4n + 2$ , respectively with  $n = 1, 2, 3, \dots$ . Basically, one finds that the corresponding Hamiltonian can be any combination of the three invariants  $X, R$  and

$|\phi_1|^2|\phi_2|^2$ . For example, the 4th order cross-gradient integrable Hamiltonian is given by

$$\begin{aligned} H = & A(\phi_1^*\phi_2 + \phi_2^*\phi_1) + i\gamma(\phi_2^*\phi_1 - \phi_1^*\phi_2) \\ & + D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)^2 + D_2|\phi_1|^2|\phi_2|^2 + D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)(|\phi_1|^2 + |\phi_2|^2) + D_4(|\phi_1|^2 + |\phi_2|^2)^2. \end{aligned} \quad (18)$$

Using Eqs. (14) it is easy to write down the corresponding equations of motion and show that only  $\dot{X} = 0$  thereby obtaining an integrable but not superintegrable cross-gradient Hamiltonian of 4th order.

Generalization to any arbitrary cross-gradient Hamiltonian of order  $4n$  is straightforward. It is easy to check that the corresponding  $4n$ 'th order cross-gradient Hamiltonian is given by

$$\begin{aligned} H = & A(\phi_1^*\phi_2 + \phi_2^*\phi_1) + i\gamma(\phi_2^*\phi_1 - \phi_1^*\phi_2) \\ & + D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{2n} + D_2(|\phi_1|^2|\phi_2|^2)^n + D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{2n-1}(|\phi_1|^2 + |\phi_2|^2) \\ & + \dots + D_{(n+1)^2}(|\phi_1|^2 + |\phi_2|^2)^{2n}, \end{aligned} \quad (19)$$

having  $(n+1)^2$  arbitrary parameters and for this case one can show that only  $\dot{X} = 0$  thereby obtaining an integrable but not superintegrable cross-gradient Hamiltonian of  $4n$ 'th order with  $(n+1)^2$  parameters.

In the same way, it is easy to check that the  $(4n+2)$ 'th order integrable Hamiltonian is given by

$$\begin{aligned} H = & A(\phi_1^*\phi_2 + \phi_2^*\phi_1) + i\gamma(\phi_2^*\phi_1 - \phi_1^*\phi_2) \\ & + D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{2n+1} + D_2(|\phi_1|^2|\phi_2|^2)^n(\phi_1^*\phi_2 + \phi_2^*\phi_1) + D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{2n}(|\phi_1|^2 + |\phi_2|^2) \\ & + \dots + D_{(n+1)(n+2)}(|\phi_1|^2 + |\phi_2|^2)^{2n+1}, \end{aligned} \quad (20)$$

having  $(n+1)(n+2)$  arbitrary parameters and for this case too only  $\dot{X} = 0$  thereby obtaining an integrable but not superintegrable cross-gradient Hamiltonian of  $(4n+2)$ 'th order.

## 5 Integrable Schrödinger Type Dimers Based on Straight-Gradient Hamiltonians

So far we have obtained integrable as well as superintegrable Schrödinger type dimer models of arbitrary even order. All these models are based on the so called cross-gradient Hamiltonians with the canonical

structure as given by Eqs. (14). It is then worth inquiring if there are integrable as well as superintegrable Schrödinger type dimer models which are based on direct-gradient Hamiltonians, i.e. in this case the canonical coordinate-momentum pairs would be  $\phi_1, \dot{\phi}_1^*$  and  $\phi_2, \dot{\phi}_2^*$  so that in this case the Hamilton's equations would be

$$i\frac{d\phi_1}{dt} = \frac{\partial H}{\partial \phi_1^*}, \quad i\frac{d\phi_2}{dt} = \frac{\partial H}{\partial \phi_2^*}. \quad (21)$$

In this connection, it is worth pointing out that recently, Barashenkov et al. [8] have presented the most general quartic integrable Schrödinger type dimer model based on the straight-gradient quartic Hamiltonian. In this model the two constants of motion are  $H$  and  $X$ . We now generalize their results and write down the most general integrable dimer models of arbitrary even order based on the straight-gradient Hamiltonians.

As shown by Barashenkov et al. [8], the most general quartic integrable straight-gradient Hamiltonian (which gives rise to cubic dimer equations) is given by

$$\begin{aligned} H = & A(\phi_1^*\phi_2 + \phi_2^*\phi_1) + B(|\phi_1|^2 + |\phi_2|^2) + i\gamma(|\phi_1|^2 - |\phi_2|^2) \\ & + D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)^2 + D_2(\phi_1^*\phi_2 + \phi_2^*\phi_1)(|\phi_1|^2 + |\phi_2|^2) + D_3(|\phi_1|^2 + |\phi_2|^2)^2. \end{aligned} \quad (22)$$

Using Eqs. (21), this gives rise to the cubic dimer equations

$$\begin{aligned} i\frac{d\phi_1}{dt} = & A\phi_1 + B\phi_2 + i\gamma\phi_1 + 2D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)\phi_2 \\ & + D_3(|\phi_1|^2 + |\phi_2|^2)\phi_2 + D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)\phi_1 + 2D_4(|\phi_1|^2 + |\phi_2|^2)\phi_1, \end{aligned} \quad (23)$$

$$\begin{aligned} i\frac{d\phi_2}{dt} = & A\phi_2 + B\phi_1 - i\gamma\phi_2 + 2D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)\phi_1 \\ & + D_3(|\phi_1|^2 + |\phi_2|^2)\phi_1 + D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)\phi_2 + 2D_4(|\phi_1|^2 + |\phi_2|^2)\phi_2. \end{aligned} \quad (24)$$

Using these field equations, it is easy to check that the variable  $X$  as defined by Eq. (15) is time independent, i.e.  $\dot{X} = 0$  and hence it is an integrable system with a straight-gradient Hamiltonian.

Generalization to an arbitrary even order is now straightforward. In particular, the most general  $2n$ 'th order straight-gradient integrable Hamiltonian is given by

$$\begin{aligned} H = & A(\phi_1^*\phi_2 + \phi_2^*\phi_1) + B(|\phi_1|^2 + |\phi_2|^2) + i\gamma(|\phi_1|^2 - |\phi_2|^2) \\ & + D_1(\phi_1^*\phi_2 + \phi_2^*\phi_1)^n + D_2(|\phi_1|^2 + |\phi_2|^2)(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{n-1} + \dots + D_{n+1}(|\phi_1|^2 + |\phi_2|^2)^n. \end{aligned} \quad (25)$$



It is easy to check that in these models also, the variable  $X$  as defined by Eq. (15) is time independent, i.e.  $\dot{X} = 0$ .

## 6 Straight-Gradient Superintegrable Quartic Schrödinger Dimer

Finally, we present a quartic straight-gradient superintegrable model. However, we must clarify that so far we have not been able to generalize these results and obtain higher order superintegrable models based on the straight-gradient Hamiltonian. Consider the Hamiltonian

$$H = i\gamma(|\phi_1|^2 - |\phi_2|^2) + A_1(\phi_1\phi_2^* + \phi_2\phi_1^*) + A_2(|\phi_1|^2 + |\phi_2|^2) + D_1(|\phi_1|^2 + |\phi_2|^2)^2 + D_2|\phi_1|^2|\phi_2|^2 + D_3(\phi_1\phi_2^* + \phi_2\phi_1^*)^2. \quad (26)$$

This gives rise to the cubic dimer equations

$$i\frac{d\phi_1}{dt} = A_1\phi_2 + A_2\phi_1 + i\gamma\phi_1 + 2D_1(|\phi_1|^2 + |\phi_2|^2)\phi_1 + D_2|\phi_2|^2\phi_1 + 2D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{n-1}\phi_2, \quad (27)$$

$$i\frac{d\phi_2}{dt} = A_1\phi_1 + A_2\phi_2 - i\gamma\phi_1 + 2D_1(|\phi_1|^2 + |\phi_2|^2)\phi_2 + D_2|\phi_1|^2\phi_2 + 2D_3(\phi_1^*\phi_2 + \phi_2^*\phi_1)^{n-1}\phi_1. \quad (28)$$

Using these field equations, it is easy to work out  $\dot{X}, \dot{Y}$  as defined by Eqs. (15). We find

$$\dot{X} = -D_2YZ, \quad \dot{Y} = 2A_1Z + (D_2 + 4D_3)XZ, \quad \dot{R} = 2\gamma Z. \quad (29)$$

It is now easily checked that the two constants of motion are  $C_1, C_2$  where

$$C_1 = (aX - b)^2 + Y^2, \quad C_2 = R + f \sin^{-1} \left( \frac{aX - b}{\sqrt{C_1}} \right), \quad (30)$$

and

$$a = \sqrt{1 + \frac{4D_3}{D_2}}, \quad b = -\frac{2A_1}{aD_2}, \quad f = \frac{2\gamma}{aD_2}. \quad (31)$$

Summarizing, we then have a superintegrable dimer model based on the straight-gradient Hamiltonian with the three constants of motion being  $H, C_1, C_2$ .

## 7 Conclusion

In recent years it has been widely realized that the PT-symmetric systems occupy a position in between the dissipative and conservative systems [1]. In this connection, the PT-symmetric dimers have been playing an insightful role in several areas [7]. It is thus of considerable importance to discover integrable and superintegrable dimers. Our work offers a step in that direction. We have presented a wide class of integrable oscillator type dimers whose Hamiltonian is of arbitrary even order. In addition, we presented a wide class of integrable and superintegrable nonlinear Schrödinger type dimers where again the Hamiltonian is of arbitrary even order.

There are several interesting open problems. For example, can one construct superintegrable oscillator type dimers? Secondly, can one consider superintegrable Schrödinger type dimers of arbitrary even order based on straight-gradient Hamiltonians? Besides, it is useful to work out the detailed dynamics of these models. Finally, it would be worthwhile knowing if some of these models could have physical relevance. We hope to address some of these issues in the near future.

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